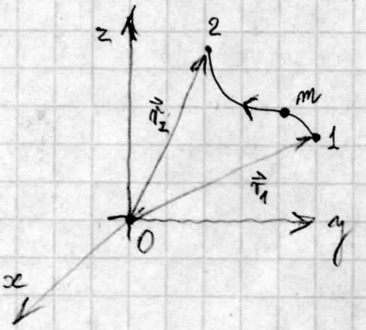
The work done by the external force and the kinetic energy

The work done by the external force $\vec{F}$ upon the particle in going from point 1 to point 2:



$$W_{12} \equiv \int_1^2 \vec{F}\cdot d\vec{r},$$ in general W_{12} depends on path between points 1 and 2;

$$\vec{F}\cdot d\vec{r} = \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt = \vec{F}\cdot\vec{v}\, dt = m\frac{d\vec{v}}{dt}\cdot\vec{v}\, dt,$$

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = m\int_{t_1}^{t_2} \frac{d\vec{v}}{dt}\cdot\vec{v}\, dt,$$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{dv}{dt}v,$$ where $v \equiv |\vec{v}|$,

$$\frac{dv}{dt}v = \frac{1}{2}\frac{d}{dt}(v^2),$$

$$W_{12} = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}(v^2)\, dt = \frac{1}{2}mv(t_2)^2 - \frac{1}{2}mv(t_1)^2,$$

$K \equiv \frac{1}{2}mv^2$ — the kinetic energy of the particle,

$$W_{12} = K_2 - K_1 \equiv \Delta K.$$

Conservative force and the potential energy

If the force field $\vec{F}$ is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force is said to be conservative.

The independence of W_{12} on the particular path between points 1 and 2 implies that the work done around any closed path is zero:

$$\oint \vec{F}\cdot d\vec{r} = 0.$$

A necessary and sufficient condition that the force $\vec{F}$ be conservative, is that $\vec{F}$ be gradient of some scalar function of position:

$$\vec{F} = -\nabla U(\vec{r}) = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right),$$

where U is called the potential energy.

We can add to U any quantity constant in space, hence the zero level of U is arbitrary.

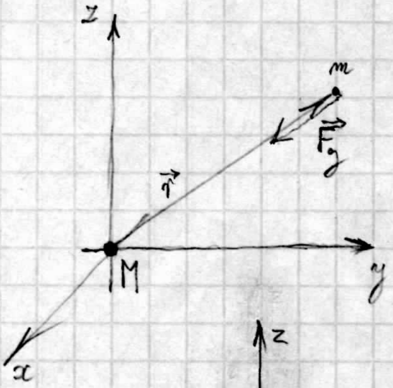
Then $\vec{F}\cdot d\vec{r} = -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = -dU$,

and $W_{12} = -\int_1^2 dU = -(U_2 - U_1) \equiv -\Delta U = \Delta K \implies \Delta U + \Delta K = 0$

or $K_1 + U_1 = K_2 + U_2$,

this is energy conservation theorem for a particle: if the forces acting on a particle are conservative, then the total energy of the particle, $K+U$, is conserved.

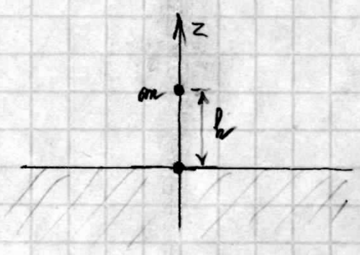
The gravitational potential energy



$$\vec{F}_g(\vec{r}) = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\vec{r},$$

$$\vec{F}_g(\vec{r}) = -\nabla U(\vec{r}), \quad U(\vec{r}) = -\frac{GMm}{r} + C \quad \left(\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}\right);$$

$$\lim_{r\to\infty} U(\vec{r}) = 0 \Rightarrow C = 0.$$



Near the surface of Earth:

$$U(h) = -\frac{GM_\oplus m}{R_\oplus + h} + C,$$

$$\lim_{h\to 0} U(h) = 0 \Rightarrow C = +\frac{GM_\oplus m}{R_\oplus},$$

$$U(h) = \frac{GM_\oplus m}{R_\oplus} - \frac{GM_\oplus m}{R_\oplus + h} = \frac{GM_\oplus m}{R_\oplus}\left(1 - \frac{1}{1+\frac{h}{R_\oplus}}\right)$$

$$= \frac{GM_\oplus m}{R_\oplus}\left(1 - \left(1 - \frac{h}{R_\oplus} + O\left((h/R_\oplus)^2\right)\right)\right)$$

$$= \frac{GM_\oplus m}{R_\oplus^2} h + O\left((h/R_\oplus)^2\right)$$

$$\cong mgh, \quad g = \frac{GM_\oplus}{R_\oplus^2} \cong 9.80\ m/s^2.$$