Principle of relativity

Newton's first law (law of inertia):

inertial reference frame (i.r.f.) exists.

I.r.f. is a frame in which any body which does not interact with other bodies stays at rest or moves with constant velocity (i.e. uniformly along a straight line).

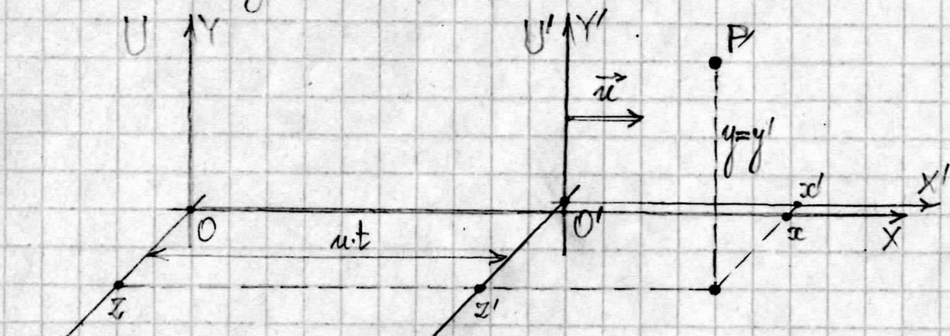
There are infinitely many i.r.f. – any reference frame which is in translational motion with constant velocity with respect to some i.r.f., is itself an i.r.f.

Principle of relativity: all i.r.f. are equivalent.

Galileo: the laws of mechanics are the same in every i.r.f.

Einstein: the laws of physics are the same in every i.r.f.

The Galilean transformation

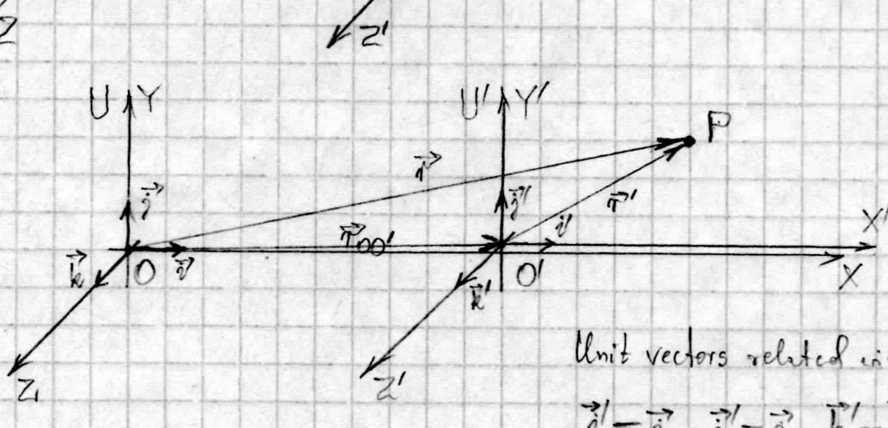


Convention: origins O and O' coincide when $t=0=t'$;

$$\begin{cases} t'=t, \\ x'=x-ut, \\ y'=y,\quad z'=z. \end{cases}$$

Inverse transformation:

$$\begin{cases} t=t', \\ x=x'+ut'=x'+ut, \\ y=y',\quad z=z'. \end{cases}$$



Unit vectors related with both coordinate systems coincide:

$$\vec{i}'=\vec{i},\quad \vec{j}'=\vec{j},\quad \vec{k}'=\vec{k};$$

$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}=[x,y,z]$, $\quad \vec{r}'=x'\vec{i}'+y'\vec{j}'+z'\vec{k}'=x'\vec{i}+y'\vec{j}+z'\vec{k}=[x',y',z']$,

$\vec{r}_{OO'}=x_{O'}\vec{i}=ut\,\vec{i}=[ut,0,0]$; hence (by virtue of inverse Galilean transformation)

$$\underline{\underline{\vec{r}=\vec{r}_{OO'}+\vec{r}'=\vec{u}\cdot t+\vec{r}'.}}$$

Transformation of velocity of a point particle,

$$\vec{v} := \left(\frac{d\vec{r}}{dt}\right)_U = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k},$$

$$v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt};$$

$$\vec{v}' := \left(\frac{d\vec{r}'}{dt}\right)_{U'} = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' = v'_{x'}\vec{i}' + v'_{y'}\vec{j}' + v'_{z'}\vec{k}',$$

$$v'_{x'} = \frac{dx'}{dt},\ v'_{y'} = \frac{dy'}{dt},\ v'_{z'} = \frac{dz'}{dt};$$

velocity of i.r.f. U' relative to i.r.f. U:

$$\vec{u} := \left(\frac{d\vec{r}_{OO'}}{dt}\right)_U = \frac{dx_{O'}}{dt}\vec{i} + 0\vec{j} + 0\vec{k} = u\vec{i}.$$

For components with respect to $(\vec{i},\vec{j},\vec{k})$: $\vec{v} = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$,

$\vec{v}' = \left[\frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt}\right]$ (because $\vec{i}'=\vec{i}$, $\vec{j}'=\vec{j}$, and $\vec{k}'=\vec{k}$),

$\vec{u} = [u, 0, 0]$.

From Galilean transformation: $x(t) = x'(t) + ut$, $y(t) = y'(t)$, $z(t) = z'(t)$,

hence $\frac{dx}{dt} = \frac{dx'}{dt} + u$, $\frac{dy}{dt} = \frac{dy'}{dt}$, $\frac{dz}{dt} = \frac{dz'}{dt}$,

then $\underline{\underline{\vec{v} = \vec{v}' + \vec{u}}}$ — the Galilean law of addition of velocities.

Transformation of acceleration of a point particle,

$$\vec{a} := \left(\frac{d^2\vec{r}}{dt^2}\right)_U = \left(\frac{d\vec{v}}{dt}\right)_U = a_x\vec{i} + a_y\vec{j} + a_z\vec{k},$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},\ a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},\ a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2};$$

$$\vec{a}' := \left(\frac{d^2\vec{r}'}{dt^2}\right)_{U'} = \left(\frac{d\vec{v}'}{dt}\right)_{U'} = a'_{x'}\vec{i}' + a'_{y'}\vec{j}' + a'_{z'}\vec{k}',$$

$$a'_{x'} = \frac{dv'_{x'}}{dt} = \frac{d^2x'}{dt^2},\ a'_{y'} = \frac{dv'_{y'}}{dt} = \frac{d^2y'}{dt^2},\ a'_{z'} = \frac{dv'_{z'}}{dt} = \frac{d^2z'}{dt^2};$$

acceleration of i.r.f. U' relative to i.r.f. U: $\left(\frac{d^2\vec{r}_{OO'}}{dt^2}\right)_U = \left(\frac{d\vec{u}}{dt}\right)_U = \vec{0}.$

From Galilean transformation: $\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}$, $\frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}$, $\frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2}$,

thus $\underline{\underline{\vec{a} = \vec{a}'}}.$

An <u>event</u> is any physical phenomenon which occurs in a very small region of space (in a point of space) and lasts very shortly. Collision of two elementary particles is a good model of an event.

Given reference frame, to any event one can assign three <u>spatial coordinates</u> x, y, z defining the location of its occurrence, and moment of time t telling when it has occured. The moment of time t we call <u>time coordinate</u> of the event.

The <u>event has an absolute character</u>, i.e. it does not depend on the choice of reference frame, but spatial and time coordinates of the event are relative — they do depend on the choice of reference frame.

The collection of all possible events is called <u>spacetime.</u>
Spacetime is four-dimensional: each event is described by three spatial coordinates and one time coordinate.

To measure time coordinate of events one needs to employ collection of clocks which are supposed to be spaced densly enough, so there is a clock next to every event of interest, ready to record its time of occurrence <u>without any delay.</u>
All clocks ticking off time coordinate t are <u>synchronized</u> and all run at the same rate.

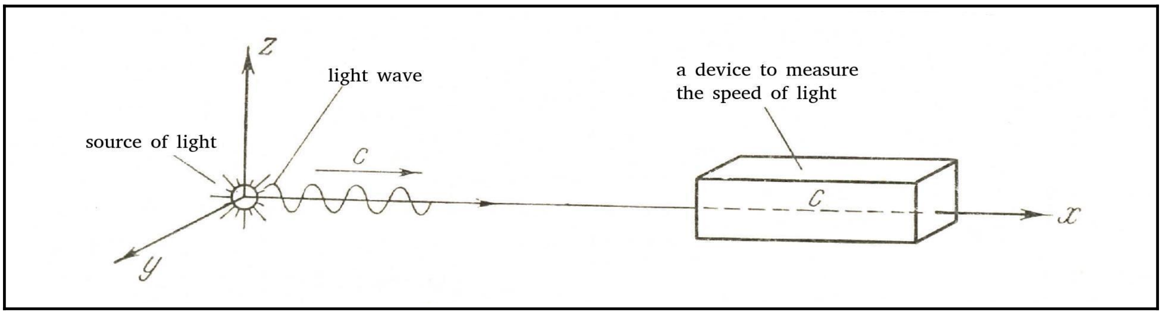
Clocks can be synchronized by means of exchanging light signals.

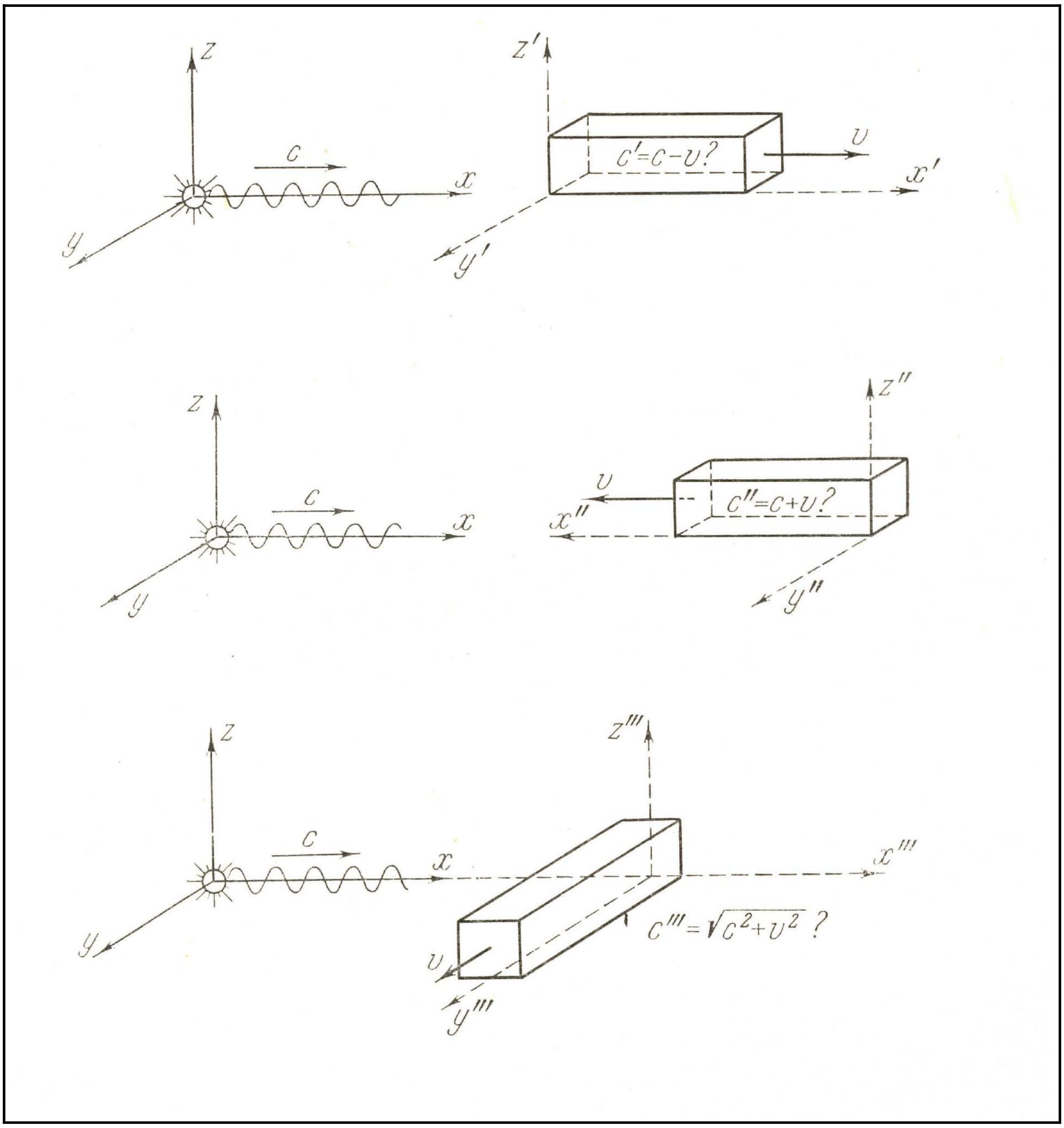
The clocks remain at rest with respect to each other.

t_A^s — clock A sends a signal,
t_A^r — the signal returns to clock A,
t_B — the signal is reflected at the location of clock B;

$$t_B - t_A^s = t_A^r - t_B \Rightarrow \underline{\underline{t_B = \tfrac{1}{2}\left(t_A^s + t_A^r\right)}}.$$

Speed of light and the non-relativistic law of addition of velocities





The non-relativistic law of addition of velocities predicts that the speeds c', c'', c''' should be different from c. Experiments show that $c' = c'' = c''' = c$.

The first experiment in which the lack of influence of the speed of reference system on the measured value of the speed of light was checked, was performed by A.A. Michelson in 1881, it was then repeated with a greater accuracy by A.A. Michelson and E.W. Morley in 1887. These experiments also show that there is no *ether*, a hypothetical medium in which electromagnetic waves would propagate.

Postulates of special relativity theory:

(i) principle of relativity,

(ii) universality of the speed of light:

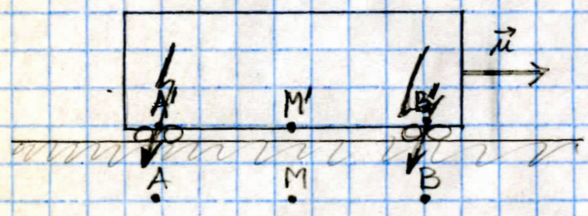
the speed of light in vacuum is the same in all i.r.f.

and is independent of the motion of the light source, its value equals

$$c = 299\,792\,458 \text{ m/s} \quad \text{(exactly)}.$$

The 2nd postulate implies that it is impossible for an inertial observer to travel at c with respect to any other inertial observer.

Relativity of simultaneity (a thought experiment).

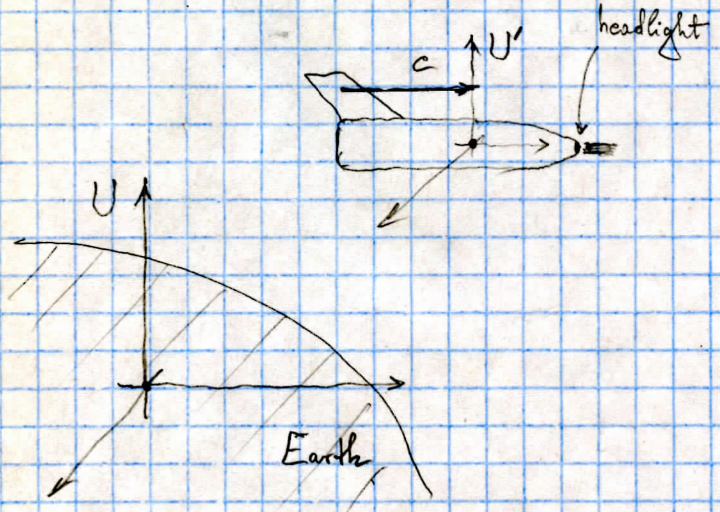


$AM = MB, \; A'M' = M'B'$

Two lightning bolts strike a car, one near each end. Each bolt leaves a mark on the car (at A' and B') and one on the ground (at A and B) at the instant the bolt hits.

Suppose the two light flashes from the lightning strikes reach observer at M simultaneously, this observer concludes that the two bolts struck A and B simultaneously.

Observer at M' is moving to the right with the train, so he runs into the light flash from B' before the light flash from A' reaches him, this observer concludes that the lightning bolt at B' struck before the one at A'.



- Suppose that the spacecraft U' is moving at c relative to an observer U on the Earth.
- If the spacecraft turns on a headlight, the 2nd postulate asserts that the observer U measures the headlight beam to be also moving at c — thus U measures that the headlight beam and the spacecraft move together.
- But the 2nd postulate also asserts that the headlight beam moves at a speed c relative to the spacecraft.
- This contradictory result can be avoided only if it is impossible for an inertial observer to move at c.

Distances measured in a direction perpendicular to the relative motion (download for free at https://openstax.org/details/books/university-physics-volume-3)

Imagine two observers moving along their *x*-axes and passing each other while holding meter sticks vertically in the *y*-direction. Figure 5.10 shows two meter sticks M and M' that are at rest in the reference frames of two boys S and S', respectively. A small paintbrush is attached to the top (the 100-cm mark) of stick M'. Suppose that S' is moving to the right at a very high speed *v* relative to S, and the sticks are oriented so that they are perpendicular, or transverse, to their relative velocity vector. The sticks are held so that as they pass each other, their lower ends (the 0-cm marks) coincide. Assume that when S looks at his stick M afterwards, he finds a line painted on it, just below the top of the stick. Because the brush is attached to the top of the other boy's stick M', S can only conclude that stick M' is less than 1.0 m long.

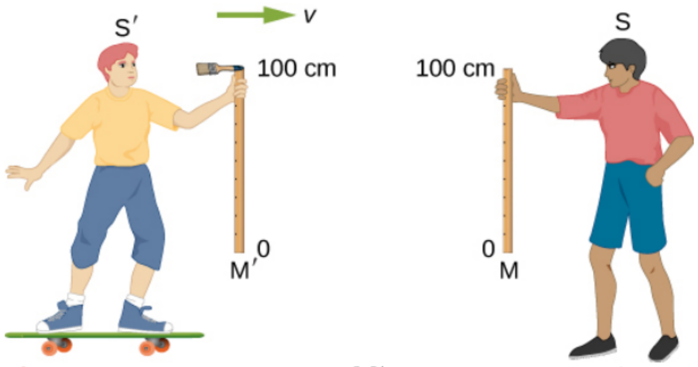
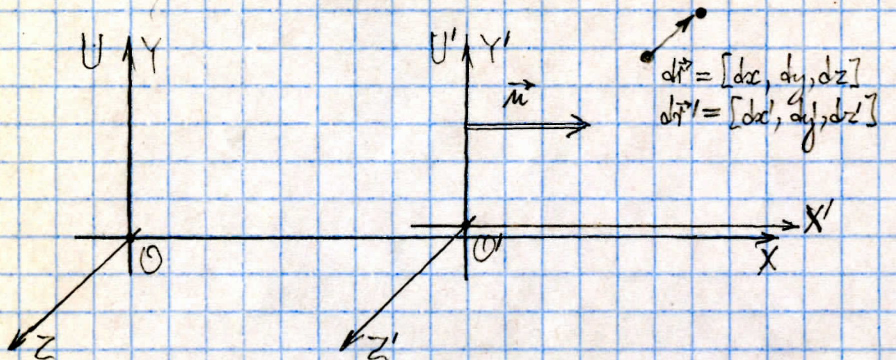


Figure 5.10 Meter sticks M and M' are stationary in the reference frames of observers S and S', respectively. As the sticks pass, a small brush attached to the 100-cm mark of M' paints a line on M.

Now when the boys approach each other, S', like S, sees a meter stick moving toward him with speed *v*. Because their situations are symmetric, each boy must make the same measurement of the stick in the other frame. So, if S measures stick M' to be less than 1.0 m long, S' must measure stick M to be also less than 1.0 m long, and S' must see his paintbrush pass over the top of stick M and not paint a line on it. In other words, after the same event, one boy sees a painted line on a stick, while the other does not see such a line on that same stick!

Einstein's first postulate requires that the laws of physics (as, for example, applied to painting) predict that S and S', who are both in inertial frames, make the same observations; that is, S and S' must either both see a line painted on stick M, or both not see that line. We are therefore forced to conclude our original assumption that S saw a line painted below the top of his stick was wrong! Instead, S finds the line painted right at the 100-cm mark on M. Then both boys will agree that a line is painted on M, and they will also agree that both sticks are exactly 1 m long. We conclude then that measurements of a transverse *length must be the same in different inertial frames*.

Lorentz transformation



Empty space is homogeneous and isotropic, time is homogeneous, therefore relations between space and time coordinates in two inertial frames should be given by linear functions:

$$\begin{cases} y' = y, \quad z' = z, \\ x' = Ax + Bt, \quad t' = Mx + Nt, \end{cases}$$

where A, B, M, N can depend on the relative velocity $\vec{u}$ of i.r.f. U' relative to i.r.f. U.

Let us consider a point particle moving with respect to both reference frames. With respect to U the particle changes its location by $d\vec{r} = [dx, dy, dz]$ in time dt, in i.r.f. U' the change is $d\vec{r}' = [dx', dy', dz']$ in time dt'. We thus have

$dx' = A\,dx + B\,dt$, $dy' = dy$, $dz' = dz$, $dt' = M\,dx + N\,dt$, hence

$$v'_{x'} = \frac{dx'}{dt'} = \frac{A\,dx + B\,dt}{M\,dx + N\,dt} = \frac{A\frac{dx}{dt} + B}{M\frac{dx}{dt} + N} = \frac{A v_x + B}{M v_x + N}, \text{ where } v_x = \frac{dx}{dt}, \text{ similarly}$$

$$v'_{y'} = \frac{v_y}{M v_x + N}, \quad v'_{z'} = \frac{v_z}{M v_x + N}.$$

- Let us consider a particle which is at rest in U', then $v'_{x'} = v'_{y'} = v'_{z'} = 0$ and $v_x = u$ $(v_y = v_z = 0)$, hence $0 = \frac{Au + B}{Mu + N} \Rightarrow B = -Au$.
- Let the particle be at rest in U, then $v_x = v_y = v_z = 0$ and $v'_{x'} = -u$ $(v'_{y'} = v'_{z'} = 0)$, hence $-u = \frac{B}{N} \Rightarrow N = -\frac{B}{u} = A$.
- Let us consider a photon moving in U' in the direction of the O'X' axis, then $v'_{x'} = c$ (and $v'_{y'} = v'_{z'} = 0$), but also $v_x = c$ (and $v_y = v_z = 0$), hence $c = \frac{Ac - Au}{Mc + A} \Rightarrow M = -\frac{u}{c^2}A$.

We thus have

$$v'_{x'} = \frac{A v_x - Au}{-\frac{u}{c^2}A v_x + A} = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}, \quad v'_{y'} = \frac{v_y}{A\left(1 - \frac{u v_x}{c^2}\right)}, \quad v'_{z'} = \frac{v_z}{A\left(1 - \frac{u v_x}{c^2}\right)}.$$

- Let us now consider the photon travelling in whichever direction in U', then $|\vec{v}'| = \sqrt{(v'_{x'})^2 + (v'_{y'})^2 + (v'_{z'})^2} = c$. Let us compute:

$$c^2 = (v'_{x'})^2 + (v'_{y'})^2 + (v'_{z'})^2 = \frac{(v_x - u)^2}{\left(1 - \frac{u v_x}{c}\right)^2} + \frac{v_y^2 + v_z^2}{A^2\left(1 - \frac{u v_x}{c}\right)^2}.$$

But $c^2 = v_x^2 + v_y^2 + v_z^2 \Rightarrow v_y^2 + v_z^2 = c^2 - v_x^2$, what we substitute to the above equation, after that we solve it with respect to A:

$$A^2 = \frac{1}{1-u^2/c^2}, \quad \text{hence} \quad A = \pm\frac{1}{\sqrt{1-u^2/c^2}}.$$

We choose the plus sign, because for $u=0$ it has to be $v'_{y'} = v_y$ and $v'_{z'} = v_z$.

Finally the Lorentzian law of addition of velocities reads

$$v'_{x'} = \frac{v_x - u}{1-\frac{uv_x}{c^2}}, \quad v'_{y'} = \frac{v_y\sqrt{1-u^2/c^2}}{1-\frac{uv_x}{c^2}}, \quad v'_{z'} = \frac{v_z\sqrt{1-u^2/c^2}}{1-\frac{uv_x}{c^2}}. \qquad (*)$$

Principle of relativity implies, that it is easy to obtain from (*) formulae for components of the velocity relative to U as functions of components of the velocity relative to U': it is enough to replace u by $-u$ and interchange primed and unprimed quantities. The result is

$$v_x = \frac{v'_{x'} + u}{1+\frac{uv'_{x'}}{c^2}}, \quad v_y = \frac{v'_{y'}\sqrt{1-u^2/c^2}}{1+\frac{uv'_{x'}}{c^2}}, \quad v_z = \frac{v'_{z'}\sqrt{1-u^2/c^2}}{1+\frac{uv'_{x'}}{c^2}}.$$

Lorentz transformation takes the form

$$x' = \frac{x - ut}{\sqrt{1-u^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{u}{c^2}x}{\sqrt{1-u^2/c^2}}. \qquad (**)$$

Making again use of the principle of relativity one can immediately obtain transformation inverse to that given by equations (**) [again by interchanging primed and unprimed quantities and replacing u by $-u$]:

$$x = \frac{x' + ut'}{\sqrt{1-u^2/c^2}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{u}{c^2}x'}{\sqrt{1-u^2/c^2}}.$$

- The Lorentz transformation and the Lorentzian law of addition of velocities ensure, that if a body moves with a constant velocity (i.e. uniformly along a straight line) with respect to some i.r.f., then it moves with a constant velocity with respect to any other i.r.f.

√

We define the <u>interval between any two events</u> Z_1 [with spacetime coordinates (t_1, x_1, y_1, z_1)] and Z_2 [with spacetime coordinates (t_2, x_2, y_2, z_2)] to be

$$\underline{I_{12} := -c^2(t_2-t_1)^2 + (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2.}$$

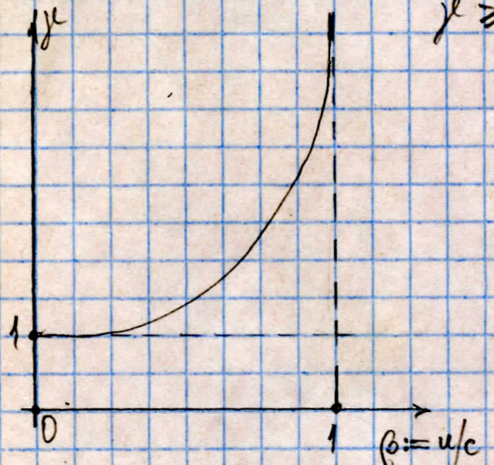
<u>The interval is invariant under Lorentz transformation:</u>
if (t_1', x_1', y_1', z_1') and (t_2', x_2', y_2', z_2') are spacetime coordinates of the events Z_1 and Z_2, respectively, with respect to any other i.r.f., then

$$-c^2(t_2'-t_1')^2 + (x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2 =$$
$$= -c^2(t_2-t_1)^2 + (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2.$$

For two events that are infinitesimally close to each other:
$t_2 = t_1 + dt, \quad x_2 = x_1 + dx, \quad y_2 = y_1 + dy, \quad z_2 = z_1 + dz,$
the interval between two such events is often denoted by ds^2, we thus have

$$\underline{ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.}$$

<u>The Lorentz factor</u> $\gamma := \dfrac{1}{\sqrt{1-u^2/c^2}}$,



$\gamma \geq 1, \quad \gamma \to \infty$ when $u \to c$.

The Lorentz transformation rewritten with the aid of γ:

$x' = \gamma(x-ut), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{u}{c^2}x\right);$

$x = \gamma(x'+ut'), \quad y = y', \quad z = z', \quad t = \gamma\left(t' + \frac{u}{c^2}x'\right).$

Here u is the component (along the OX axis) of the vector of velocity $\vec{u}$ of i.r.f. U' with respect to i.r.f. U, hence

$$-c < u < c;$$

the negative values of u $(u<0)$ mean that i.r.f. U' moves relative to U in the direction opposite to the direction of OX axis.

x

Let a particle moves with respect to i.r.f. U with velocity

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} = [v_x, v_y, v_z],$$

the same particle relative to some other i.r.f. U′ has velocity

$$\vec{v}' = v'_{x'} \vec{i}' + v'_{y'} \vec{j}' + v'_{z'} \vec{k}' = [v'_{x'}, v'_{y'}, v'_{z'}].$$

The Lorentzian law of addition of velocities gives the relations

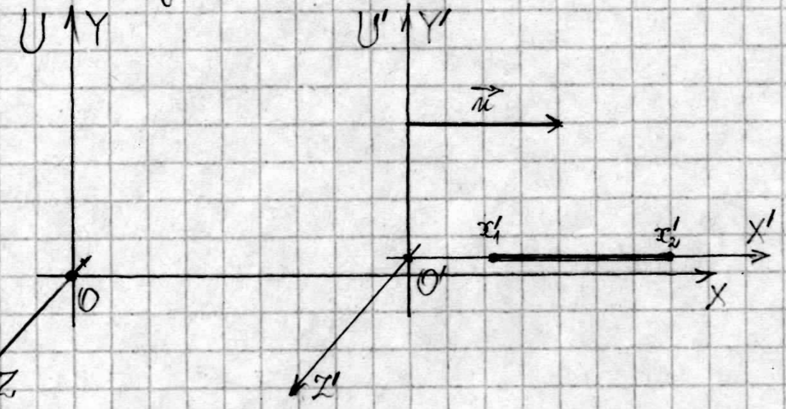
$$v'_{x'} = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}, \quad v'_{y'} = \frac{v_y}{\gamma\left(1 - \frac{u v_x}{c^2}\right)}, \quad v'_{z'} = \frac{v_z}{\gamma\left(1 - \frac{u v_x}{c^2}\right)},$$

which imply the following facts:

- if $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = c$, then $|\vec{v}'| = \sqrt{(v'_{x'})^2 + (v'_{y'})^2 + (v'_{z'})^2} = c$;
- if $|\vec{v}| < c$, then $|\vec{v}'| < c$.

√

Length contraction



A ruler is at rest in i.r.f. U',
it lies on the axis O'X' of this frame.
The length of the ruler in U' equals
$$l_0 = x_2' - x_1'.$$
What length of the ruler does the observer related with U measure?

To measure the length of a <u>moving</u> ruler, one has to measure the coordinates x_1, x_2 of the ruler's both ends <u>at the same time</u> (i.e., simultaneously).
Then the length of the moving ruler equals:
$$l = x_2 - x_1, \quad \text{provided } t_1 = t_2.$$

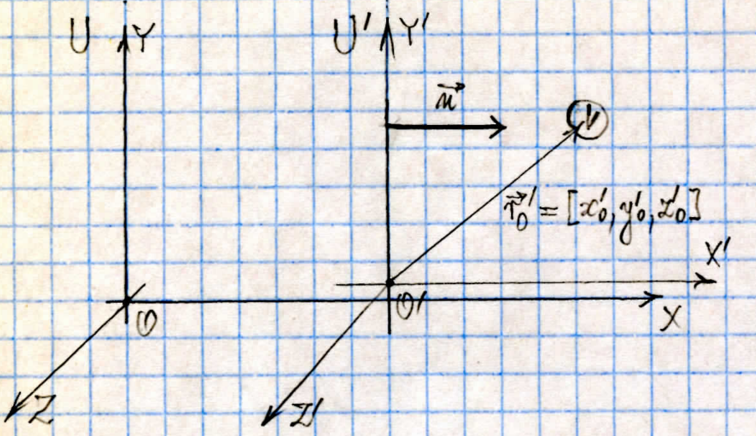
From Lorentz transformation:
$$l_0 = x_2' - x_1' = \gamma(x_2 - ut_2) - \gamma(x_1 - ut_1)$$
$$= \gamma(x_2 - x_1) - u\gamma(t_2 - t_1) = \gamma(x_2 - x_1), \quad \text{because } t_1 = t_2,$$

hence $l_0 = \gamma l$, that is $\underline{l = \frac{1}{\gamma} l_0 = \sqrt{1 - \frac{u^2}{c^2}}\, l_0.}$

$\gamma \geq 1 \Rightarrow l \leq l_0$ — <u>length contraction</u> ($l < l_0$ provided $u \neq 0$).

A length measured in the frame in which the body is at rest (the rest frame of the body) is called a <u>proper length</u> of the body.
The length of the body measured in any frame moving relative to the rest frame of the body is shorter than the proper length of the body.

Time dilation



Let us consider two events, which occur in a fixed point of space relative to i.r.f. U', space coordinates of the point are x_0', y_0', z_0. Time interval between these two events measured by a clock at rest at the point (x_0', y_0', z_0) equals

$$\tau_0 = t_2' - t_1'.$$

What time interval between these events does an observer related with i.r.f. U measure?

From Lorentz transformation:

$$\tau = t_2 - t_1 = \gamma\left(t_2' + \frac{u}{c^2}x_2'\right) - \gamma\left(t_1' + \frac{u}{c^2}x_1'\right)$$
$$= \gamma\,(t_2' - t_1') + \frac{u}{c^2}\gamma\,(x_2' - x_1') = \gamma\,\tau_0, \quad \text{because } x_2' = x_1' = x_0',$$

hence $\underline{\underline{\tau = \gamma\,\tau_0}}$, thus $\tau \geqslant \tau_0$ — time dilation:

an observer measures any clock to run slow if it moves relative to him.

Time measured by clocks at rest with respect to some object we call the <u>proper time</u> of the object.

Causality in special relativity

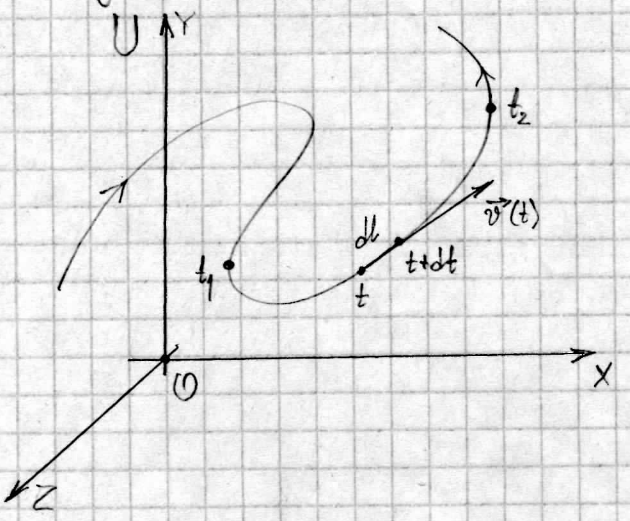
Let us consider two events with spacetime coordinates (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2);
one of these events can be the <u>cause</u> for the other (the <u>effect</u>) if and only if

$$l_{12} \leq c\,t_{12},$$

where $l_{12} := \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$ and $t_{12} := t_2 - t_1$.

Let us aditionally assume that $t_2 > t_1$ (event 1 is the couse for event 2), then in any other i.r.f. $t_2' > t_1'$: <u>special relativity respects causality.</u>

Proper time



A clock is moving (in an arbitrary way) with respect to IRF U. Between the moments of time t and $t+dt$, the clock has covered a distance

$$dl = \sqrt{dx^2 + dy^2 + dz^2},$$

an instantaneous velocity of the clock at time t, $\vec{v}(t)$, has had magnitude $v = |\vec{v}| = dl/dt$.

Let us compute time interval dt' measured by the clock, which corresponds to the interval dt registered by clocks staying at rest in IRF U.

Let us consider an <u>IRF U' momentarily comoving with the clock at time t</u>, then $dx' = dy' = dz' = 0$, invariance of the spacetime interval implies that

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -c^2 dt'^2,$$

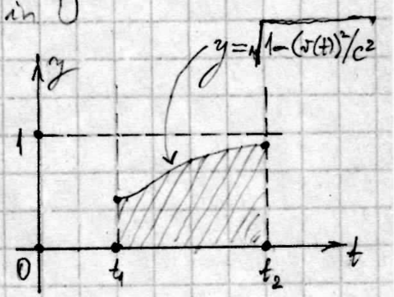
$$-c^2 dt^2 + dl^2 = -c^2 dt'^2 \implies dt' = \pm\sqrt{1 - \frac{1}{c^2}\frac{dl^2}{dt^2}}\, dt = \pm\sqrt{1 - \frac{v^2}{c^2}}\, dt;$$

we choose the + sign:

$$dt' = +\sqrt{1 - \frac{v^2}{c^2}}\, dt.$$

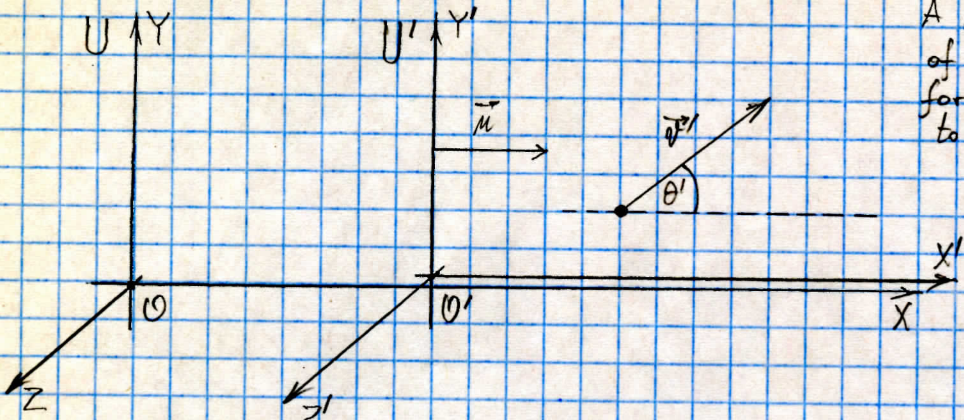
Time interval $\langle t_1; t_2 \rangle$ registered by clocks at rest in U corresponds to proper-time interval $\langle t_1'; t_2' \rangle$:

$$t_2' - t_1' = \int_{t_1}^{t_2} \sqrt{1 - \frac{(v(t))^2}{c^2}}\, dt.$$



If $\vec{v}(t) \not\equiv \vec{0}$, then $t_2' - t_1' < t_2 - t_1$ — time dilation.

Aberration



A particle moves in the $(X'Y')$ plane of IRF U', its velocity vector $\vec{v}'$ forms an angle θ' with respect to the axis X'.

What angle θ the velocity vector $\vec{v}$ of the particle with respect to IRF U forms with the axis X?

$v'_{x'} = v'\cos\theta'$, $\quad v'_{y'} = v'\sin\theta'$, $\quad v'_{z'} = 0$, then

$$v_x = v\cos\theta = \frac{v'_{x'}+u}{1+\frac{uv'_{x'}}{c^2}} = \frac{v'\cos\theta'+u}{1+\frac{uv'}{c^2}\cos\theta'},$$

$$v_y = v\sin\theta = \frac{v'_{y'}}{\gamma\left(1+\frac{uv'_{x'}}{c^2}\right)} = \frac{v'\sin\theta'}{\gamma\left(1+\frac{uv'}{c^2}\cos\theta'\right)}, \qquad v_z = 0.$$

Hence $\underline{\operatorname{tg}\theta = \frac{\sin\theta}{\cos\theta} = \frac{v_y}{v_x} = \frac{v'\sin\theta'}{\gamma(u+v'\cos\theta')}}$.

Let us consider now the change of direction of a light ray (photon):

$$v' = c, \quad \operatorname{tg}\theta = \frac{c\sin\theta'}{\gamma(u+c\cos\theta')} = \sqrt{1-\frac{u^2}{c^2}}\,\frac{\sin\theta'}{\frac{u}{c}+\cos\theta'}.$$

Let $\frac{u}{c} \ll 1$, then $\operatorname{tg}\theta = \operatorname{tg}\theta' - \frac{\sin\theta'}{\cos^2\theta'}\frac{u}{c} + O\left(\left(\frac{u}{c}\right)^2\right)$;

aberration angle: $\Delta\theta := \theta' - \theta$,

for $\frac{u}{c} \ll 1$ the angle $\Delta\theta$ is small, then we can expand $\operatorname{tg}\theta = \operatorname{tg}(\theta'-\Delta\theta)$ in a series with respect to $\Delta\theta$: $\operatorname{tg}\theta = \operatorname{tg}\theta' - \frac{1}{\cos^2\theta'}\Delta\theta + O\left((\Delta\theta)^2\right)$.

Neglecting higher-order terms we thus have:

$$\operatorname{tg}\theta' - \frac{1}{\cos^2\theta'}\Delta\theta \cong \operatorname{tg}\theta' - \frac{\sin\theta'}{\cos^2\theta'}\frac{u}{c},$$

hence $\underline{\Delta\theta \cong \frac{u}{c}\sin\theta'}$.