Spacetime diagrams

It is convenient to introduce time coordinate with dimension of length:

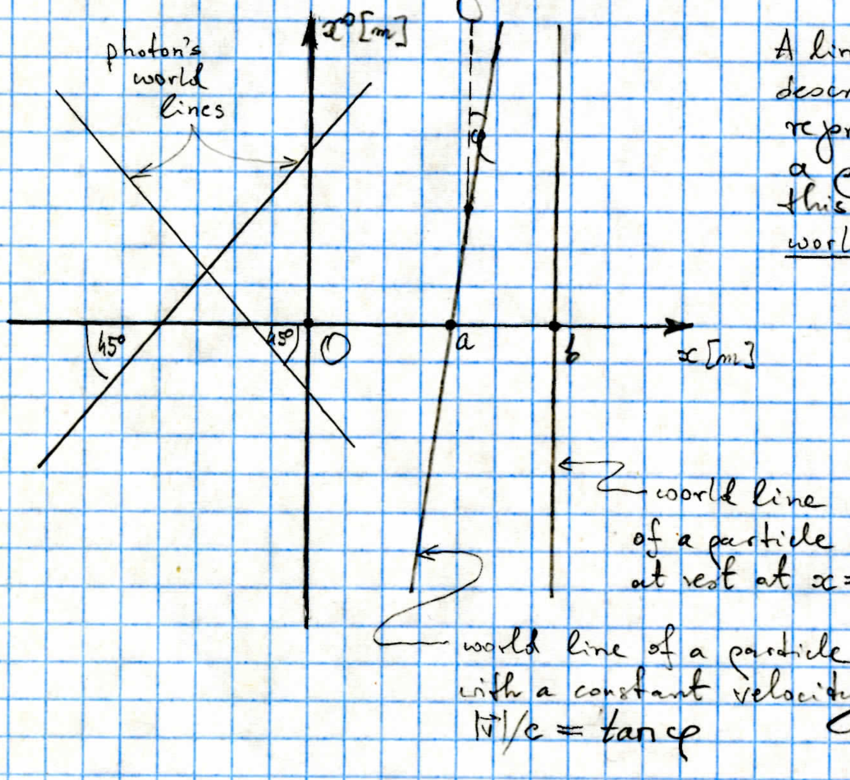
$x^0 := c\,t$ — conventional time coordinate (measured e.g. in seconds); x^0: upper index; c: speed of light

If e.g. $x^0 = 1\,m$, then $t = \frac{x^0}{c} \cong \frac{1\,m}{3\times10^8\,m/s} \cong 3.3\times10^{-9}\,s = 3.3\,ns.$

In given IRF each event has thus four spacetime coordinates: $x^0 = ct,\ x,\ y,\ z.$

We will mostly study 2-dimensional slices of spacetime, that is we will restrict ourselves to study events with y and z coordinates fixed (and usually with $y=z=0$, i.e. we will consider motions along x axis).

2-dimensional slice of spacetime we can describe by means of Cartesian coordinate system with axes x^0 and x.



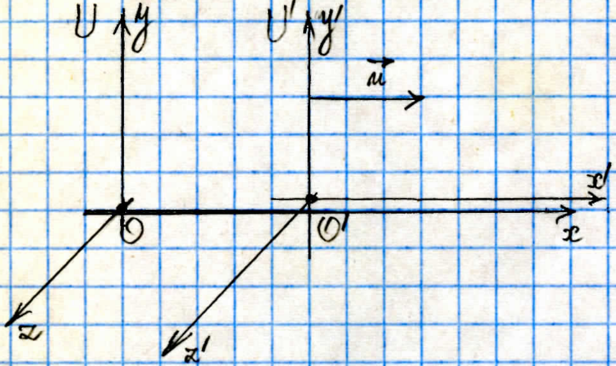
A line in the (x^0, x) plane described by a relation $x = x(x^0)$ represents the position of a particle at different times, this is called the particle's <u>world line</u>.

World line of a particle moving with constant velocity:

$$
\begin{aligned}
x &= a + v_0 t \\
&= a + (v_0/c)\,ct \\
&= a + (v_0/c)\,x^0 \\
&= a + \tan\varphi\, x^0, \qquad \text{with } \tan\varphi = v_0/c,
\end{aligned}
$$

so this is the straight line with the slope φ (with respect to the x^0 axis).

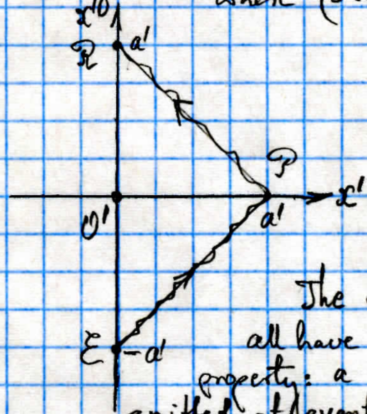
Construction of coordinates used by another observer



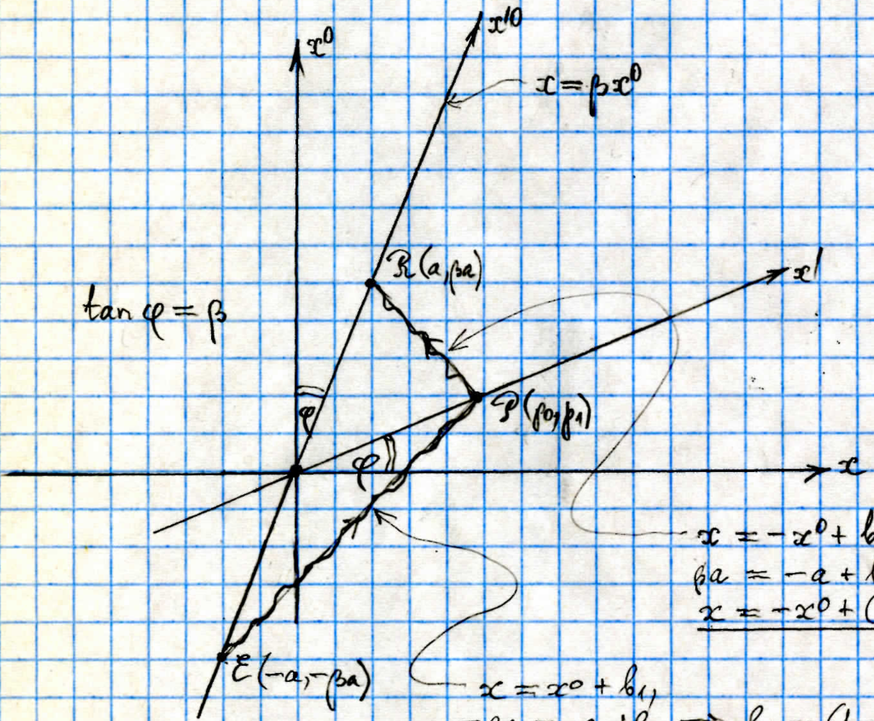
The x'^0 axis coincides with the world line of the origin O' of IRF U', it has thus the following equation in IRF U:

$$\underline{x} = ut = \frac{u}{c}ct = \frac{u}{c}x^0 = \underline{\beta x^0},$$

where $\beta := u/c$.



The events on x' axis all have the following property: a light ray emitted at event $\mathcal{E}$ with $x'=0$ and $x'^0 = -a'$ will reach the x' axis at $x'=a'$ (we call this event $\mathcal{P}$); if reflected, it will return to the point $x'=0$ at $x'^0 = +a'$, called event $\mathcal{R}$.



$x = -x^0 + b_2,$

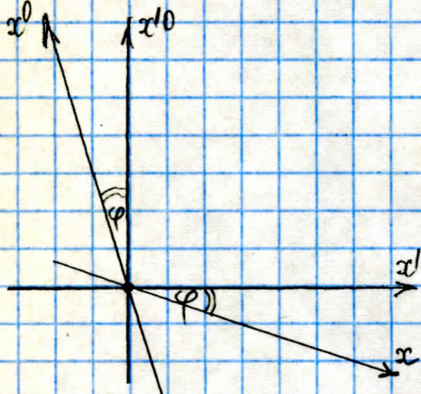
$\beta a = -a + b_2 \Rightarrow b_2 = (1+\beta)a,$

$\underline{x = -x^0 + (1+\beta)a}$

$x = x^0 + b_1,$

$-\beta a = -a + b_1 \Rightarrow b_1 = (1-\beta)a$

$\underline{x = x^0 + (1-\beta)a}$



$$\begin{cases} p_1 = -p_0 + (1+\beta)a \\ p_1 = p_0 + (1-\beta)a \end{cases} \Longrightarrow \begin{cases} p_0 = \beta a \\ p_1 = a \end{cases},$$

hence the x' axis is described by equation $\underline{x = \frac{1}{\beta}x^0}$

$(x = k\, x^0 \Rightarrow a = k\cdot\beta a \Rightarrow k = 1/\beta).$

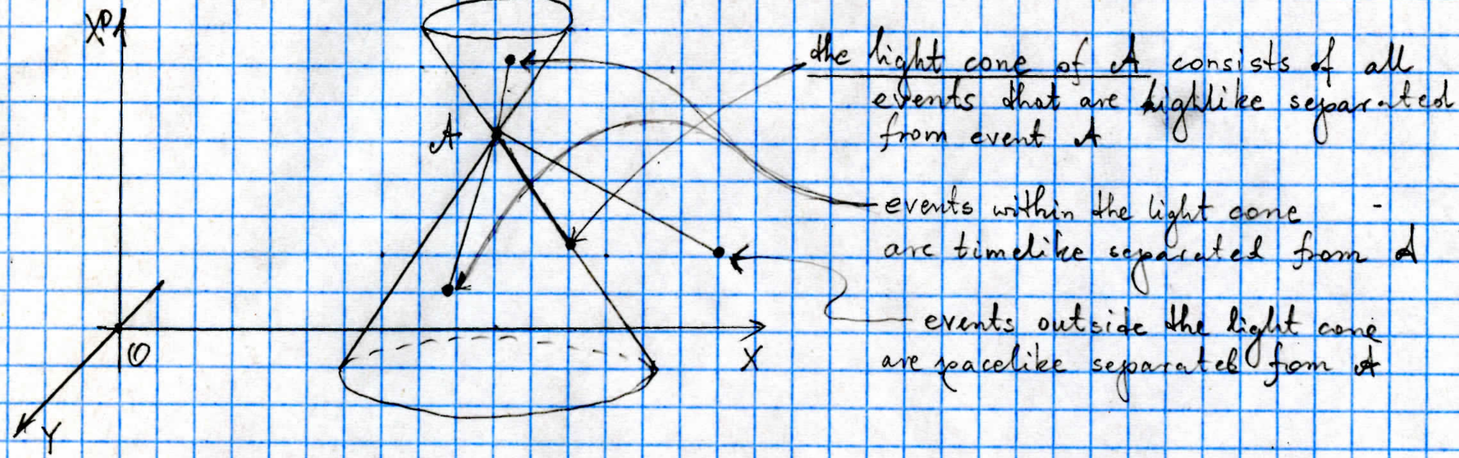
The interval between events $\mathcal{I}_1$ (with coordinates x^0_1, x_1, y_1, z_1) and $\mathcal{I}_2$ (with coordinates x^0_2, x_2, y_2, z_2) equals

$$I_{12} = -(x^0_2 - x^0_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

If $I_{12} > 0$, the events are said to be <u>spacelike separated</u>,

if $I_{12} < 0$, the events are said to be <u>timelike separated</u>,

if $I_{12} = 0$, the events are said to be <u>lightlike</u> or <u>null separated</u>.



Let the event A has spacetime coordinates (x^0_A, x_A, y_A, z_A), then <u>the light cone of A</u> is described by equation

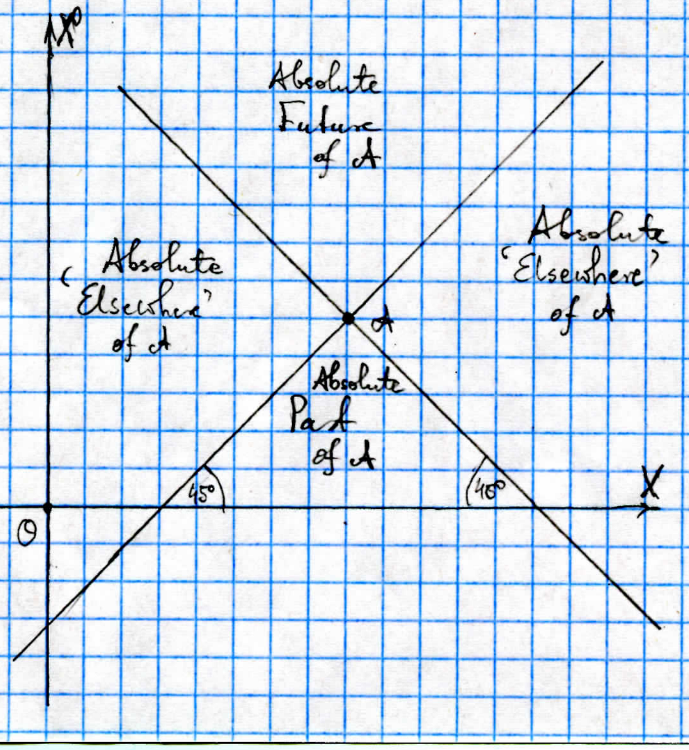
$$-(x^0 - x^0_A)^2 + (x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 = 0;$$

<u>interior of the light cone</u> is described by inequality

$$-(x^0 - x^0_A)^2 + (x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 < 0,$$

and <u>exterior of the light cone</u> is defined by inequality

$$-(x^0 - x^0_A)^2 + (x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 > 0.$$



Let us consider the light cone of the event $\mathcal{O} = (0,0,0,0)$:

$$-(x^0)^2 + x^2 + y^2 + z^2 = 0;$$

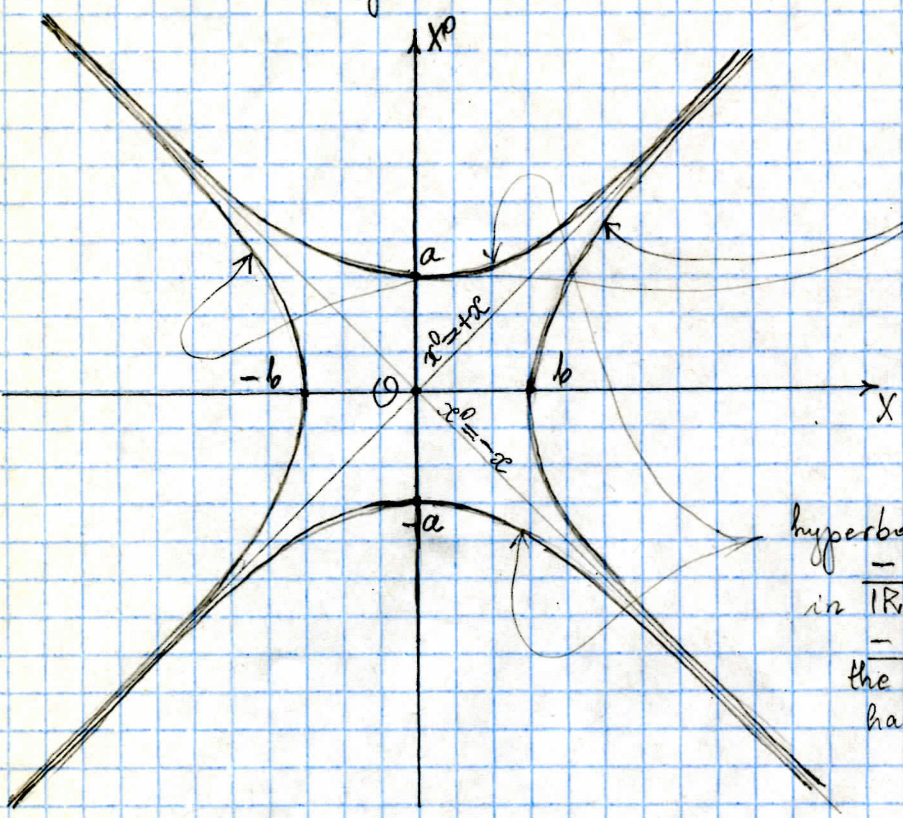
(i) $y = z = 0$ section: $(-x^0)^2 + x^2 = 0 \Leftrightarrow (-x^0 + x)(x^0 + x) = 0$

$\Leftrightarrow \underline{x^0 = +x \quad \text{or} \quad x^0 = -x};$

(ii) $z = 0$ section: $\underline{x^2 + y^2 = (x^0)^2}$ — equation of the circle in the (x,y) plane with center at $(0,0)$ and radius $|x^0|$;

(iii) $x^0 = a = \text{const}$ section: $\underline{x^2 + y^2 + z^2 = a^2}$ — equation of the sphere with center at $(0,0,0)$ and radius $|a|$.

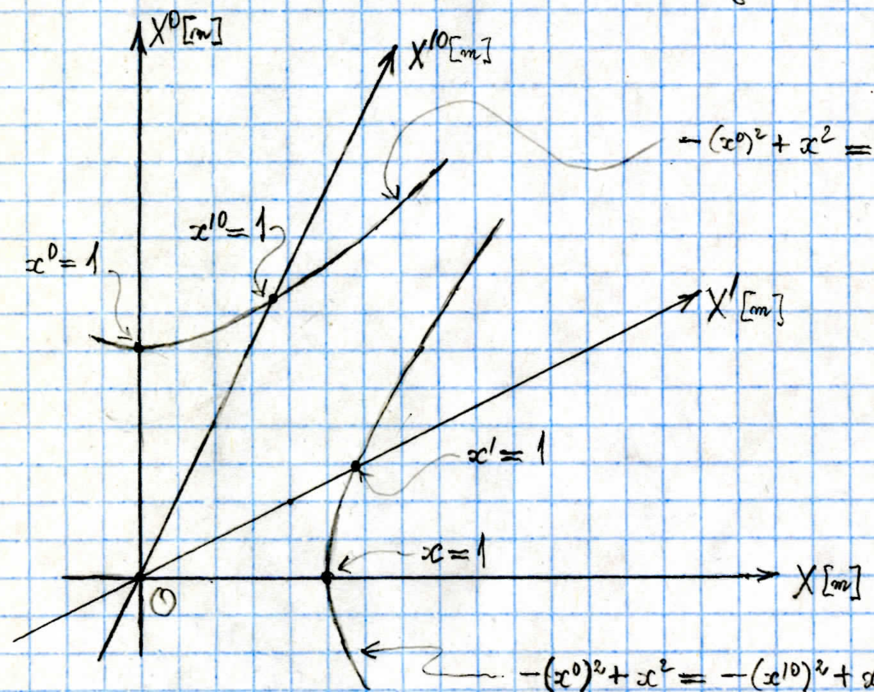
Invariant hyperbolae



hyperbola with the equation
$-(x^0)^2+x^2=b^2$ $(b>0)$,
in IRF U' it has the equation
$-(x'^0)^2+x'^2=b^2$;
it passes through all events
whose (spacelike) interval
from the origin O is b^2

hyperbola with the equation
$-(x^0)^2+x^2=-a^2$ $(a>0)$,
in IRF U' it has the equation
$-(x'^0)^2+x'^2=-a^2$;
the events on this hyperbola all
have (timelike) interval $-a^2$
from the origin O

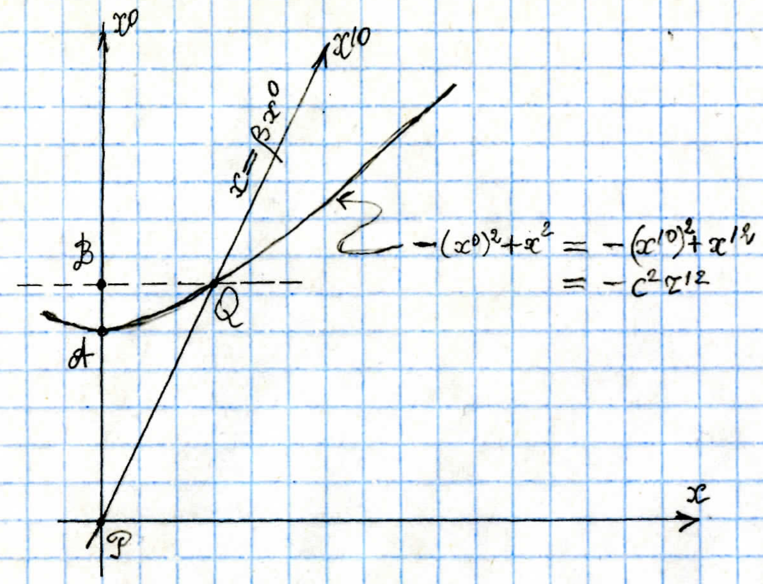
Calibration of the axes of U' on the diagram of the observer U



$-(x^0)^2+x^2=-(x'^0)^2+x'^2=-1$

$-(x^0)^2+x^2=-(x'^0)^2+x'^2=+1$

Time dilation and length contraction without Lorentz transformation



$Q(c\tau', 0)$ in U', $I_{\mathcal{P}Q} = -c^2\tau'^2$,

$\mathcal{A}(c\tau', 0)$ in U, $\mathcal{B}(c\tau, 0)$ in U,

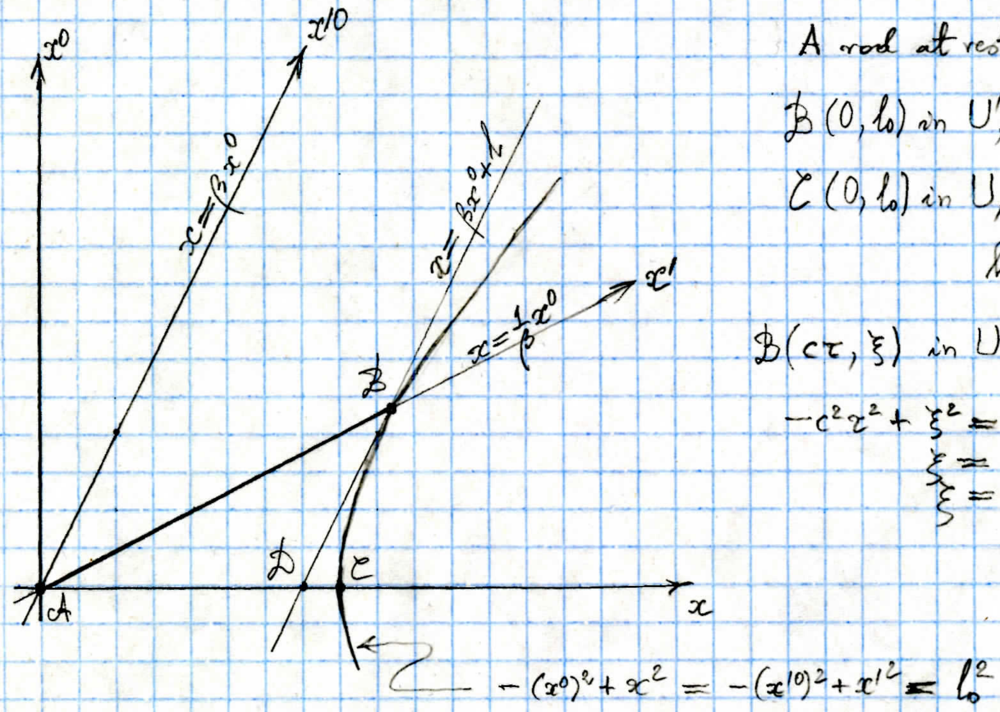
hence $c\tau > c\tau'$,

$\tau > \tau'$.

$Q(c\tau, \xi)$ in U,

$$-c^2\tau^2 + \xi^2 = -c^2\tau'^2$$
$$\xi = \beta c\tau$$

$$\tau = \frac{\tau'}{\sqrt{1-\beta^2}}.$$



A rod at rest in U',

$\mathcal{B}(0, l_0)$ in U', $I_{\mathcal{A}\mathcal{B}} = l_0^2$,

$\mathcal{C}(0, l_0)$ in U, $\mathcal{D}(0, l)$ in U,

hence $l < l_0$.

$\mathcal{B}(c\tau, \xi)$ in U,

$$-c^2\tau^2 + \xi^2 = l_0^2$$
$$\xi = {}^1\!/_\beta\,(c\tau)$$
$$\xi = \beta c\tau + l$$

$$l = \sqrt{1-\beta^2}\, l_0.$$